The First Soviet-German Workshop
"Digital Image Processing"
Berlin, November 5-9, 1990

Enumeration of Topological Sortings of Acyclic Digraphs

Oleg G. Taraszow

Institute of Computer Architecture
in Central Institute of Cybernetics
and Information Processes
Kurstraße 33, O-1086 Berlin

ABSTRACT

We study the problem of the determination of the number $\tau(G)$ of all different topological sortings of an acyclic digraphs G. It is shown that the original problem can be reduced to the case of directed trees by the application successively of the proposed method for the removal of cycles. The answer for directed trees is given in an analytical form. Two families of upper bounds are derived for the number $\tau(G)$, too. The first one originates from the set of spanning trees of G, and the second one is based on the set of minimal Dilworth decompositions of G. It is shown that the first family of upper bounds is tight.

1. Introduction.

We study the determination of such invariant of a finite acyclic digraph as the number of its topological sortings.

Let G = (V,E) be a finite, acyclic digraph with sets V(G) and E(G) of vertices and edges, respectively, and $n := |V(G)|$, $\mathbb{N}_n := \{1,2,\ldots,n\}$.

A <u>topological sorting</u> of G is a mapping $\varphi : \mathbb{N}_n \longrightarrow V(G)$ such that holds

$$(\forall\, i,j \in \mathbb{N}_n) : (\varphi(i),\varphi(j)) \in E(G) \Longrightarrow i < j. \tag{1}$$

Problem 1 (enumeration of topological sortings)

GIVEN: A finite, acyclic digraph G with n vertices.

FIND: The number $\tau(G)$ of topological sortings of G.

Now we present three another problems that are equivalent to the problem 1 (for more details see Taraszow [1-2]).

A <u>tournament</u> is a complete, oriented graph.

A <u>supertournament</u> of an acyclic digraph G is a tournament such that the digraph G is a spanning subdigraph of this tournament.

Problem 2 (enumeration of supertournaments)

GIVEN: A finite, acyclic digraph G with n vertices.

FIND: The number $\tau(G)$ of supertournaments of G.

Let $R \subseteq X \times X$ be a <u>partial order relation</u> defined on the set X with n distinct elements,i.e.

(a) $R \cap E = E$ - <u>reflexivity</u>,

(b) $R \cap R^{-1} = \emptyset$ - <u>anti-symmetry</u>,

(c) $R^2 \subseteq R$ - <u>transitivity</u>.

A partial order $R \subseteq X \times X$ is a <u>linear order</u> iff holds

$$R \cup R^{-1} = X^2.$$

A <u>linear extension</u> of R is a linear order $L \subseteq X \times X$ such that holds $R \subseteq L$.

Problem 3 (enumeration of linear extensions)

GIVEN: A partial order $R \subseteq X \times X$ on the set X with n distinct elements.

FIND: The number $\tau(R)$ of linear extensions of R.

Let $X = \square_n$ and $R \subseteq \square_n \times \square_n$ a strict partial order, i.e. $R \cap E = \emptyset$, $R \cap R^{-1} = \emptyset$, and $R^2 \subseteq R$. A <u>permutation</u> $\psi : \square_n \longrightarrow \square_n$ is called <u>admissible</u> concerning R iff holds

$$\psi(i)R\psi(j) \Longrightarrow i < j.$$

Problem 4 (enumeration of admissible permutations)

GIVEN: A strict partial order $R \subseteq \square_n \times \square_n$.

FIND: The number $\tau(R)$ of admissible permutations concerning R.

2. RESULTS.

In this session we give some results for the determination of the number of topological sortings of a finite, acyclic digraph. These results were obtained for different problem formulations, but are given here in corresponding reformulations for the problem 1.

Let G be a finite, acyclic digraph G with

$$G = \bigcup_{i=1}^{n} G_i, \tag{2}$$

where $G_i (i = 1,2,...n)$ are weakly connected components of G.

Theorem 1.

$$\tau\left[\bigcup_{i=1}^{n} G_i\right] = |V(G)|! \prod_{i=1}^{n} \frac{\tau(G_i)}{|V(G_i)|!} \tag{3}$$

For the **proof** see Tanaev and Skurba [1].■

Let $\bar{G}$ be the transitive closure and $\hat{G}$ the basis graph of G, respectively.

Lemma 1.

$$(\bar{G}_1 = \bar{G}_2) \ \& \ (\hat{G}_1 = \hat{G}_2) \Longrightarrow \tau(G_1) = \tau(G_2). \tag{4}$$

For the **proof** see Sidorenko [2]. ■

Let $r \in E(G)$. We denote by $G-r$ and $G/\bar{r}$ the digraphs resulting from G by canceling the arc r and by changing the orientation of the arc r, respectively.

Lemma 2. $\forall\, r \in E(G)$: $\tau(G) = \tau(G-r) + \tau(G/\bar{r})$. (5)

For the **proof** see Taraszow [3]. ■

Let $od_G(x)$ and $id_G(x)$ denote the outdegree and the indegree of a vertex $x \in V(G)$ in a graph G, respectively. For a circuit L of G ($L \subseteq G$) the vertices $b \in V(G)$ and $e \in V(G)$, respectively, are initial and terminal vertices of L iff $id_L(b) = 0$ and $od_L(e) = 0$, respectively.

Let L be a simple circuit of an acyclic digraph G. Now we label the arcs of L as follows. Starting with any terminal vertex

$e \in V(L)$ we label counterclockwise with $1,2,\dots,m$ all clockwise oriented arcs of L. Analogously, starting with the terminal vertex $e \in V(L)$ we label clockwise with $-1,-2,\dots,-n$ all counterclockwise oriented arcs of L. Here m and n are the numbers of clockwise and counterclockwise oriented arcs, respectively.

Let L_i^* be the circuit resulting from L by changing the orientation of $|i|$ arcs of L labeled with $1,2,\dots,i$ if $i \geq 0$ and with $-1,-2,\dots,i$ if $i \leq 0$. We denote then by L_i the digraph originating from the circuit L_i^* by eliminating the arc with the label i. The digraph G/L_i results from G by exchanging of L for L_i.

Theorem 2 (circuits elimination)

For any simple circuit L (if L exists) of a finite, acyclic digraph G holds

$$\tau(G) = \frac{1}{2} \sum_{i=-n}^{m} (-1)^{i+1} \tau(G/L_i). \tag{6}$$

For the **proof** see Taraszow [3]. ■

Theorem 3. For any finite, acyclic digraph G the problem of the determination of $\tau(G)$ can be reduced to the case of finite, directed, rooted trees using successively the circuits elimination.

For the **proof** see Taraszow [4]. ■

Let T be a finite, directed, rooted tree and T_x denote for any $x \in V(T)$ the maximal subtree in T with the root x.

Theorem 4.

$$\tau(T) = \frac{|V(T)|\,!}{\prod_{x \in V(T)} |V(T_x)|} \tag{7}$$

For the **proof** see Taraszow [3]. ■

Let $T_{p,q}$ be a finite, rooted, regular, oriented tree of a degree p and a level q ($p,q \in \mathbb{N}$)

Corollary.

$$\tau(T_{p,q}) = (p-1)^{p\frac{p^q-1}{p-1}} \left[p\,\frac{p^q-1}{p-1}\right]! \Big/ \prod_{i=1}^{q} (p^{q-i+1}-1)^{p^i}. \tag{8}$$

Let T[G] denote an oriented, rooted, spanning tree of a weakly connected digraph G.

Theorem 5. Let G be a finite, acyclic, weakly connected digraph with exactly one initial vertex and $T[\hat{G}]$ be the such oriented, rooted, spanning tree of G that holds $\hat{G} = T[G]$. Then

$$\tau(G) = \frac{|V(G)|\,!}{\prod_{x \in V(T)} |V(T_x[G])|} \tag{9}$$

For the **proof** see Taraszow [3]. ■

3. Two families of upper bounds.

In this session we give two analytical families of upper bounds for $\tau(G)$. The first one originates from the set of spanning trees of G while the second one is based on the set of minimal Dilworth decompositions of G. If G is a directed tree, the first family of upper bounds provides the exact number $\tau(G)$.

Lemma 3.

$$(G_* \leq G \leq G^*) \wedge (V(G_*) = V(G) = V(G^*)) \Longrightarrow \tau(G^*) \leq \tau(G) \leq \tau(G_*).$$

For the **proof** see Taraszow [4]. ■

Theorem 6. Let G be a finite, labeled, acyclic, weakly connected digraph with exactly one initial vertex. Then for any its rooted spanning tree holds

$$\tau(G) \leq \frac{|V(G)|\,!}{\prod_{x \in V(T)} |V(T_x[G])|} \tag{10}$$

For the **proof** see Taraszow [4]. ■

Let N(G) denote the set of independent sets of a digraph G, and q(G) the vertex number of maximal independent sets of G, i.e.

$$N(G) := \{ X \subseteq V(G) \mid (\forall x,y \in X)((x,y) \notin E(G)) \wedge ((y,x) \notin E(G))\},$$

and

$$q(G) = \max_{X \in N(G)} |X|.$$

A <u>minimal</u> <u>Dilworth</u> <u>decomposition</u> of an acyclic digraph G is a such set $\mathfrak{D} = \{C_1, C_2, ..., C_{q(G)}\}$ of paths $C_i (i = 1,2,...,q(G))$ of $\bar{G}$ that holds

$$V(G) = \bigcup_{i=1}^{q(G)} V(C_i) \quad \text{and} \quad \forall\, i \neq j : V(C_i) \cap V(C_j) = \emptyset.$$

Theorem 7. For an arbitrary minimal Dilworth decomposition $\mathfrak{D}=\{C_1, C_2, ..., C_{q(G)}\}$ of a finite, acyclic, weakly connected digraph with exactly one initial $b \in V(C_p)$ holds

$$\tau(G) \leq \frac{|V(G) - 1)|!}{|V(C_p) - 1)|!} \Big/ \prod_{\substack{i=1 \\ i \neq p}}^{q(G)} |V(C_i)|! \tag{11}$$

For the **proof** see Taraszow [3]. ■

REFERENCES.

[1] В.С.Танаев, В.В.Шкурба. Введение в теорию расписаний. Наука, Москва (1975).

[2] А.Ф.Сидоренко. Число допустимых линейных упорядочений частично упорядоченного множества как функция его графа несравнимости. Мат. заметки 29 (1981) 1, 75 - 82.

[3] O.G.Taraszow. Zur Anzahl der kreisfreien Turniere, die Obergraph eines endlichen markierten kreisfreien Digraphen sind. EIK 19 (1983) 10/11, 503 - 527.

[4] O.G.Taraszow. Entwicklung graphentheoretischer Methoden zur Untersuchung von Systemstrukturen. Diss. B, Berlin (1985).