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ON THE ACCURACY OF THE INPUT DATA IN TRAJECTORY PROBLEMS†

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The accuracy of the input data in trajectory problems for which the initial formulation is still physically meaningful is investigated. Various sufficient conditions are given under which the problem is well-posed in terms of the radius of stability of the matrix of input data. Trajectory problems in which all that is known is the intervals in which the elements of the distance matrix lie are investigated in the same vein.

IT IS COMMON knowledge that the initial data in discrete optimization problems are always known only with a certain error, the nature of which depends on the particular problem and is governed by a whole series of physical and economic factors. In fact, this is true of any real models, and not only discrete extremal problems. A special feature of discrete optimization problems is the unpredictability of the behaviour of their solutions when there are perturbations of the initial data, which calls for extra care when preparing for numerical solution. A fairly detailed and convincing argument on this subject is given in [1]. We shall therefore take this as a condition of the problem and focus entirely on the consequences of doing so.

Following good practice, we shall use the travelling salesman (TS) problem as our model and only discuss possible generalizations at the end of the paper.

Suppose we are given a complete n-vertex directed graph G_n with weight matrix $A = \|r_{ij}\|$ and a set of Hamiltonian cycles (HC) $\tau_1 \ldots \tau_{(n-1)!}$. The length of the HC τ_i is computed from the formula

$$\tau_i(A) = \sum_{(sk)\in\tau_i} r_{sk}. \tag{1}$$

It is required to find the optimum HC (OHC), i.e. the HC of minimum length. The set of indices of all OHC will be denoted by $\varphi(A)$. We then assume that, owing to measurement errors, rather than the "true" weight matrix A, we have to deal with the perturbed matrix $A' = \|r_{ij}'\|$, where

$$r_{ij}' = r_{ij} + \delta_{ij}. \tag{2}$$

Thus, we assume that the noise is additive in character and defined by the noise matrix $A_\delta = \|\delta_{ij}\|$. The Chebyshev norm of the matrix A_δ, that is, $\|A_\delta\| = \max\limits_{(i,j)} |\delta_{ij}|$, will be called the degree of perturbation or noise level. Clearly, if the noise level is high enough, the matrix A' can be a matrix with any prescribed optimal solution, in which case the original formulation of the problem is ill-posed, simply being meaningless. There is therefore a limit to the measurement error for which the original formulation still makes sense. Obtaining the fullest possible information about this limit, while allowing for a variety of formulations, is another aim of this paper.

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Note that the travelling salesman problem is stable in the ordinary terms of mathematical analysis, since small changes in the initial data (the elements r_{ij}) produce small changes in the value of the functional (1), or the optimal solution. At the same time it is not the actual value of the quality functional (1) on the optimal solution that is of basic interest in the travelling-salesman problem, but the form of the optimal solution, the optimal Hamiltonian cycles. With this concept of stability, it makes sense to class the TS problem as unstable [1] although, of course, everything depends on the original weight matrix.

Going on to a precise formulation of the problem, we consider several versions.

I. The formulation of the problem will be taken to be well-posed if the initial and perturbed problems have at least one optimal Hamiltonian cycle, that is $\varphi(A)\cap\varphi(A')\neq\varnothing$.

Let $\rho_0(A)$ be the radius of stability of the matrix A (cf. [2]).

Assertion 1. If $\|A_\delta\|<\rho_0(A')$, the TC problem is well-posed.

Proof. It follows from (2) and the definition of the radius of stability that if $|\delta_{ij}<\rho_0(A')$, then the matrix A lies inside the sphere of stability of the matrix A', guaranteeing that the original problem was well-posed.

Observe that the following formulation would appear to be absolutely correct. For a given weight matrix $A=\|r_{ij}\|$, it is required to determine the limiting noise level at which, for any perturbed matrix A', the condition $\varphi(A')\cap\varphi(A)\neq\varnothing$ holds.

The answer in that case would be as follows: the limiting noise level is equal to the radius of stability of the matrix A (cf. [3]). However, we only know the matrix A' in this situation, and so must be restricted by the sufficient condition given by the foregoing assertion.

Assertion 1 uses minimum information about the type of distortions of the original weight matrix, namely, its degree of perturbation. Presumably, more detailed information about the matrix would yield more useful assertions about the well-posedness of the TS problem.

By way of example, consider the case where one element r_{sk} of the weight matrix $A=\|r_{ij}\|$ is unstable. According to [4], the radius of stability of the matrix $A'=\|r_{ij}'\|$ can be found as follows.

1. In the graph G_n with weight matrix A', let there be optimal Hamiltonian cycles which contain the arc (s, k) and optimal Hamiltonian cycles which do not contain the arc (s, k). In that case $\rho_0(A')=\infty$. In fact, if the weight of the arc r_{sk}' is increased, any of the optimal Hamiltonian cycles of the old graph which do not gain the arc (s, k) will be optimal in the new graph. But if the weight of the arc r_{sk}' decreases, those Hamiltonian cycles which contain the edge (s, k) remain optimal.

2. However, if all the optimal Hamiltonian cycles of the graph G_n with weight matrix $A'=\|r_{ij}'\|$ contain the arc (s, k), then the radius of stability can be computed with the formula:

$$\rho_0(A')=\min_{\substack{j\notin\varphi(A)\\(sk)\notin\tau_j}}[\tau_j(A')-\tau_{\min}(A')] \tag{3}$$

where $\tau_{\min}(A')$ is the length of the optimal Hamiltonian cycle in the matrix A', and the minimum in (3) is taken over all non-optimal Hamiltonian cycles which do not contain the arc (s, k). This analysis can be used as follows to elucidate whether the original formulation was well-posed.

Assertion 2. If the distribution of optimal Hamiltonian cycles in graph G_n with weight matrix A' satisfies paragraph 1 above, the original problem is well-posed, whatever the noise level. But if the distribution of optimal Hamiltonian cycles satisfies paragraph 2, the original problem is well-posed provided that $\|A_\delta\|<\rho_0(A')$, where $\rho_0(A')$ is computed using (3).

The above analysis can be extended considerably to any unstable subsets of arcs of the graph G_n, in which case it is appropriate to use results about the radius of stability from [3] and [4].

II. The following definition of the well-posedness of the TS problem is associated with small changes of the optimal solution.

Definition 1. The symmetric difference of the sets τ_i and τ_j is the distance $r(\tau_i, \tau_j)$ between Hamiltonian cycles τ_i and τ_j of the graph G_n.

The HC τ_i and τ_j will be said to be close if $r(\tau_i, \tau_j)\leq k$.

The number k is determined independently and forms part of the conditions of the problem.

Definition 2. The TS problem is said to be well-posed if the perturbed weight matrix A' in graph G_n contains an OHC which is close to at least one of the OHC of the original problem.

Let $\rho_k(A')$ be the radius of k-stability of the matrix A', given by the following formula [3]:

$$\rho_k(A') = \min_{j \notin \varphi_k(A')} \max_{i \in \varphi(A')} \frac{\tau_j(A') - \tau_i(A')}{2[n - |\tau_i \cap \tau_j|]}.$$

Here $\varphi_k(A')$ is the set of HC close to the set $\varphi(A)$, that is, $p \in \varphi_k(A')$ if the set $\varphi(A)$ contains an index q such that $r(\tau_p, \tau_q) \le k$. As for Assertion 1, the following criterion can be proposed for well-posedness of the TC problem in the sense of the above definition.

Assertion 3. If $\|A_\delta\| < \rho_k(A')$, then the TS problem is well-posed.

III. The following version is associated with the consideration of explicit uncertainty, when for each element r_{ij} of the weight matrix $A = \|r_{ij}\|$ only the interval $[u_{ij}, v_{ij}]$ in which this element is situated is known. The problem in this form is sometimes called the interval TS problem. As in the previous cases, there are several possible different approaches to the concept of a solution.

Definition 3. Any matrix $A^0 = \|r_{ij}^0\|$ whose elements satisfy inequalities

$$u_{ij} \le r_{ij}^0 \le v_{ij}, \tag{4}$$

is called a realization of an interval TS problem.

We denote the set of matrices whose elements satisfy inequalities (4) by T_n.

In the case of an interval TS problem, the search for realizations with the "longest" possible solution and for those with the largest value of the functional [1] can be justified.

As before, we shall base the concept of well-posedness on the actual form of the optimal solution. Let

$$S = \bigcup_{A \in T_n} \varphi(A).$$

It is clear that the set S contains the indices of those HC which can be solutions of certain TS problems with weight matrices from the set T_n.

Definition 4. The interval TS problem (4) is said to be well-posed if the set S does not contain the indices of all HC of the complete graph G_n.

The meaning of this definition is that if $S = \{1, 2, \ldots, (n-1)!\}$, then in the absence of any further information, an arbitrary HC of the graph G_n can be taken, with equal justification, as the solution of the interval TS problem. In that case, it is reasonable to assume that the original interval formulation is inadequately defined.

It is obvious that determining whether a particular interval TS problem is well-posed involves the construction of a set S. Prescribing this construction in the form given above does not in itself comprise a constructive algorithm for synthesizing S. The problem can be solved as follows.

We will show that it is sufficient to confine ourselves to a finite number of terms in the formula for S. Consider the realization of the interval problem (4) that is furthest "to the left", that is, the matrix $A = \|u_{ij}\|$. Let $\varphi(A)$, as usual, be the set of indices of OHC of a graph with weight matrix A. In order to find out whether a given index $j \notin \varphi(A)$ belongs to set S, we proceed as follows:

(1) we find the "distance" Δ^j between the matrix A and the cone F^j from the formula

$$\Delta^j = \max_{i \in \varphi(A)} \frac{\tau_j(A) - \tau_i(A)}{n - |\tau_i \cap \tau_j|}; \tag{5}$$

(2) we construct the matrix $B = \|B_{sr}\|$ as follows:

$$b_{sr} = \begin{cases} u_{sr} + \Delta^j & \text{for } (s, r) \notin \tau_j, \\ u_{sr} & \text{for } (s, r) \in \tau_j; \end{cases}$$

(3) we "curtail" the matrix $B = \|b_{sr}\|$ according to the relations

$$b_{sr}' = \begin{cases} b_{sr}, & \text{if } b_{sr} \le v_{sr}, \\ v_{sr}, & \text{if } b_{sr} > v_{sr}; \end{cases}$$

(4) if $j \in \varphi(B')$, where $B' = \|b_{sr}'\|$, then $j \in S$, but if $j \notin \varphi(B')$, then $j \notin S$.

All the concepts used in the algorithm and a justification for its well-posedness are actually contained in [3, 4] or are easily obtained from them. To make it clear, we shall merely formulate a simple assertion on which the proposed algorithm is essentially based. Let $j \notin \varphi(A)$ and suppose that it is necessary to obtain the nearest matrix B to A, the set of OHC of which already contains the HC τ_j. Then (allowing for the fact that the elements of A can only be increased in accordance with the condition imposed on the interval TS problem) all the elements in the HC τ_j of the matrix A must be left unchanged, while the others must be increased by the same amount, determined by the condition of the transformation or, in fact, by formula (5). We conclude by considering some simple criteria for whether the interval TS problem (4) is or is not well-posed.

Let

$$b_{ij} = \frac{v_{ij} + u_{ij}}{2}, \quad b_{ij}^0 = \frac{v_{ij} - u_{ij}}{2}, \quad B^0 = \|b_{ij}^0\|, \quad \Delta^0 = \min_{i=j} b_{ij}, \quad B = \|b_{ij}\|.$$

Consider the following system of linear equations;

$$x_i + y_j = b_{ij}, \quad i \neq j, \quad i, j = 1, 2, \ldots, n. \tag{6}$$

Let Δ be the discrepancy of system (6), that is,

$$\Delta = \min_{x_i y_j} \max_{(i,j)} |x_i + y_j - b_{ij}|.$$

Assertion 4. If $\Delta^0 > \Delta$, then the interval TS problem (4) is ill-posed.

The proof of this assertion follows at once from the definition of the discrepancy and the well-known fact that all the HC in a graph G_n with weight matrix $A^0 = \|x_i + y_j\|$ are optimal, that is, $\varphi(A^0) = \{1, 2, \ldots, (n-1)!\}$.

In the most meaningful sense, Assertion 4 imposes explicit constraints on the length of the intervals of variation of weights in an interval TS problem.

Note too that finding the discrepancy of system (6) can easily be reduced to solving a linear programming problem.

The following sufficient condition for well-posedness of an interval TS problem is also a simple consequence of the concepts used above.

Assertion 5. If $b_{ij}^0 \le \rho_0(B)$, then the interval TS problem (4) is well-posed.

We note in conclusion that all the above assertions can easily be extended to trajectory problems for which the corresponding facts referring to the radius of stability hold [3].

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