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The Steiner problem: a survey*

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Abstract — For the past decade the Steiner minimal tree problem has attracted the attention of researchers in discrete optimization. A brief survey of the main results concerning the properties and algorithms of the Steiner problem in the Euclidean plane, the Steiner problem in the plane with rectilinear metric and the Steiner problem in networks is done in this paper. The main attention is paid to the recent results concerning the last problem.

1. INTRODUCTION

At present some different problems united by the term Steiner problem (SP) are investigated in discrete optimization, computational geometry, various tracing problems in circuit design, communication networks, mechanical and electrical systems, etc. The publications in these regions can be separated into two parts. In the first part the problems related to the classical Steiner problem and generalizations are considered, the main attention is paid to investigations of its properties and algorithms for solution. The second part consists of the publications involving problems which 'distantly' link with the classical Steiner problem, or applied problems which arise as a result of attempts to use the algorithms to solve the Steiner problem and in which the main attention is paid to the special features of these algorithms in concrete situations.

Two main reasons which initiated this work should be mentioned. Among the problems of discrete optimization the maximum amount of researches in 80's was dedicated to the SP and a certain progress in the area of its solution was achieved because of importance of its applications. At the same time in our country this problem is insufficiently known.

Up to now there have been two surveys [105, 161] related to the Steiner problem in networks (SPN). A brief survey of heuristics in the Steiner problem with rectilinear metric (RST) was done in [120]. A survey of some results is presented in [11]. That is why in this paper the main attention will be given to the SP in the plane, for the SPN we will give a list of the main results and attention will be paid to the latest publications which were not mentioned in the previous surveys. At the same time, taking the little accessibility of these surveys into account, we attempt to give an integral picture of the results available presently in this field.

The Steiner problem in the Euclidean plane (SPE) is concerned with the construction of the shortest network spanning a set of given points $A = \{A_1, \dots, A_n\}$ called terminals. But, in contrast to the well-known minimal spanning tree problem, in the Steiner problem it is allowed to introduce any number of additional edge interconnections called Steiner points (S-points) anywhere in the plane. The shortest possible network (which has to be a tree) is called the Steiner minimal tree (SMT) and any tree covering the set of terminals and possibly some S-points is called a Steiner tree (ST).

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The SP is apparently one of the oldest optimization problems in mathematics. It goes back to Fermat, who considered the case $n = 3$ [55], but the first solution of this case was obtained by Cavalieri and Torricelli [139]. At the beginning of the last century Steiner also considered this problem and his name was connected with this problem, probably, thanks to Courant and Robbins, who in their well-known book [36] gave a formulation and some properties of this problem is the general form. Previously, in 1934, this formulation for an arbitrary n was given by Jarnik and Kossler [85].

A popular discussion of some problems related to the Steiner problem is given by Bern and Graham [11]. The history of the Steiner problem is presented in [11, 98].

If the rectilinear metric substitutes for the Euclidean one, the rectilinear Steiner problem (RSP) is obtained. In this case, the ST covers terminals using only vertical and horizontal lines, and the distance between two points with coordinates (x_i, y_i), (x_j, y_j) is equal to $|x_i - x_j| + |y_i - y_j|$. The rectilinear Steiner tree with minimal length (RMST) is required.

The separation of this case and particular attention to it can be explained by its applied aspects, for example, in wire layout problems in circuit design.

Restrictions on the number of S-points and on the location lead to the third and the most studied statement of the problem: the Steiner problem in networks (SPN). Let a graph $G = (V, E)$ be given and let a set of vertices V consist of a set A of terminals and a set S of S-points.

In what follows $|V| = p = n + s$, $|A| = n$, $|S| = s$, $|E| = m$. In this case for an undirected network $G = (V, E, c)$ with p vertices, m edges, a cost function $c\colon E \to R$ and $A \subsetneq V$, we look for a subnetwork G_A, which spans all vertices of A and the sum of its edge costs has the minimum value. This subgraph is called the minimal Steiner tree (SMT).

These three statements will be referred to as the main statements of the problem, in distinction to all the others, which, as a rule, are either their special cases or generalizations.

The article is organized as follows. The complexity of different statements of the SP is discussed in Section 2. A survey of the Steiner problem in the Euclidean plane (SPE) is given in Section 3. The rectilinear case is discussed in Section 4. A survey of SPN is given in Section 5.

The Steiner problem has provoked the question about the ratio of the MST length and the minimal spanning tree length, which is presented in the recently solved conjecture of Gilbert–Pollak, formulated in 1968, and other less known combinatorial questions. These problems are discussed in the last section.

2. THE COMPUTATIONAL COMPLEXITY OF DIFFERENT STATEMENTS OF THE STEINER PROBLEM

At present there are no polynomial time algorithms for the main statements of the problem.

In his well-known paper [87] Karp proved that SPN is *NP*-complete. He reduced the *NP*-complete problem 'Exact cover' to SPN. This proof was one of the first among similar proofs of *NP*-completeness of discrete optimization problems.

If $s = 0$, then the SPN is equivalent to the minimal spanning tree problem, for which the well-known polynomial algorithms exist, for example, Kruskal's algorithm of complexity $O(n^2)$ [96], Prim's algorithm of complexity $O(m \log n)$ [114], and a number of recently obtained more efficient algorithms.

If $n = 2$, the SPN is equivalent to the shortest path problem between a fixed pair of graph vertices for which a polynomial algorithm also exists, for example, Dijkstra's algorithm [39] of complexity $O(n^2)$. But the SPN remains *NP*-complete even in the cases where the edge costs are of the same values, the graph is planar, the graph is bipartite and the sets A and S are its parts.

However, it was established that the SPN can be solved in polynomial time for some classes of graphs which we consider in Section 5.

Here it should be mentioned that the SPN can be solved in linear time for the class of the so-called series-parallel graphs (without subgraphs which are homeomorphic to K_4). Richey and Parker [122] consider a closely related problem (using the term SP). Let subsets of vertices $S_1, \dots, S_k$ be fixed in a graph G. Is there a partition of the graph edges into k subsets $E_1, \dots, E_k$, $E_i \cap E_j = \varnothing$, $i \neq j$, such that for any $i = 1, \dots, k$ the subset E_i forms a connected graph which covers all vertices from S_i? This problem is called the Steiner subgraph problem and its *NP*-completeness is proved even for the case of the series-parallel graphs. This interesting result characterizes indirectly this class of graphs and complexity of the SPN.

Other results related to complexity of the SPN on some particular classes of graphs are given in [109, 111].

A case of the SPN of applied interest arises in connection with Darwin's theory of evolution. A special class of trees describing the hereditary relationships, which are constructed on the basis of the differences in their DNA codes, arises here. Such trees are called phylogenetic trees. The Steiner problem in phylogeny is investigated in [59]. This problem has the following mathematical statement: the Hamming distance is taken for a set of words whose length is equal to N in a fixed alphabet T and the SP for this metric space is to find the SMT for a given $X \subseteq T^N$. It is proved that this problem is *NP*-complete even for the case where the alphabet consists of two elements. The known *NP*-complete problem 'Exact 3-cover' is reduced to it.

The *NP*-completeness of the RSP was proved in [63]. The authors reduce the known *NP*-complete problem 'Node cover in a planar graph' into another restricted node cover problem. A planar graph $G = (V, E)$ with no vertex degree exceeding 4 and an integer k are given. Is there a node cover V^* for G with $|V^*| \leq k$ and such that the subgraph of G induced by V^* is connected? The last problem is transformed into the RSP.

In [64] *NP*-completeness of the SPE was proved by transforming 'Exact 3-cover' problem to a variant of the SPE.

Some classes of the SPE and RSP which can be polynomially solved are considered in the following sections.

3. THE STEINER PROBLEM IN THE EUCLIDEAN PLANE

The statement of the problem was presented in Introduction. The first method of its solution for $n = 3$ is given in [39]. Courant and Robbins [36] along with the statement of the problem for an arbitrary n gave two fundamental properties: the number of S-points in the SMT does not exceed $n - 2$, and at any Steiner point exactly three lines meet making angles of 120° with each other. Nevertheless these two facts by themselves are not sufficient for the construction of a finite procedure of searching the SMT, since for any pair of points D, B the set of points P such that $\angle DPB = 120°$ is infinite.

The first finite procedure for the problem was suggested by Melzak [108] in 1961. He made the key observation that if two points D and B are directly connected to a Steiner point P, then the third line segment incident to P passes, being extended, through the third vertex C of an equilateral triangle with D and B as its other two vertices and lying in the half plane determined by the line passing through D and B which does not contain P. In fact, $DP + BP = CP$. Thus, the Steiner minimal tree for a set of n points in the plane, with D and B as two vertices immediately connected to a Steiner point, can be found by replacing D and B with C and solving the SP for the $n-1$ points. There are two choices for C, and $n(n-1)/2$ choices for the pair D, B. Furthermore, the SMT needs not to have as many as $n-2$ Steiner points (it is called a full Steiner tree (FST) if it does). If it does not, the SMT will be decomposed into a number of smaller FST's. If we blindly organize a computation to try all possibilities, their number will grow exponentially. These facts have helped for the construction of an exponential algorithm for the SMT.

The problem of finding an exact solution of the SPE was developed by Cockayne in some papers [29–34].

The proof of the main result in [108], presented by Melzak, contains errors and the algorithm is not clearly described. Cockayne in [29] gives a correct proof and studies the properties of the SMT both for the case of the Euclidean plane and for some other metrics. The proof based on these investigations is more perfect in comparison with the direct realization of Melzak's idea.

The first characterization of the MST was also given by Melzak in [108]. In particular, U is a MST if and only if

(1) U has the vertices $A_1, \dots, A_n, S_1, \dots, S_k$;

(2) U is not self-intersecting;

(3) vertices S_i, $i = 1, \dots, k$, are S-points and lie in the triangles formed by terminal points $A_1, \dots, A_n$;

(4) the degrees of these S-points are equal to 3 and for the terminal points, do not exceed 3;

(5) $0 \le k \le n-2$.

Some elementary properties of the MST, an algorithm to solve the SP for $n = 3$ and for some simple configurations with n points are presented by Kelmans [88].

From the listed properties of the SMT it follows, in particular, that no two edges in the SMT meet at angle less then 120°, and edges incident to S-points meet at 120°. This fact was an argument for the investigations of Hwang and Wang in [80], where the SP in the hexagonal coordinate system was discussed.

A full Steiner tree (FST) is a Steiner tree which satisfies the first four properties mentioned above and for which $k = n-2$. The construction of the minimal length FST (MFST) is an essential step in the algorithm proposed by Cockayne [29], since any SMT is a union of full Steiner subtrees. Description of the structure of MFST and a method of its enumeration serve as the basis of Cockayne's algorithm.

The use of computer in generating the MFST on a set of points in the Euclidean plane, where the terminals are vertices of a convex polygon, is demonstrated in [31]. The numerical results are supplied with a FORTRAN program.

Let W be a subset of terminals containing m vertices and $n \ge 2$. The FST spanning W has $m-2$ S-points. The adjacency matrix or any equivalent description of the tree of terminals and $m-2$ S-points is called the full topology (FT). It should be noted that an FT does not specify the locations of its S-points.

The next after the above mentioned papers of Melzak and Cockayne significant contribution in the investigations of SPE is Gilbert and Pollak's paper [66]. It is proved that if an FST for a given FT exists, then it is unique. Let $f(s)$ be the number of FT's with exactly s S-points and $F(n,s)$ be the number of topologies for trees which have n terminals and s S-points. In [66] it is shown that

$$f(s) = \frac{(2s)!}{2^s s!}, \qquad F(n,s) = \frac{(s+2)(n+s-2)!}{2^s s!}.$$

The total number of full topologies is equal to

$$\sum_{k=0}^{n-2} \sum_{m=0}^{(n-k-2)/2} \binom{n}{m} \binom{n-m}{k+m+2} f(k+m) \frac{(n+k-2)!\,(m+k)!}{(2k+2m+1)!}.$$

The authors discussed the problem of reducing an original SPE to some problems with smaller number of terminals. Seven different properties of ST, which are the basis of this reduction, were presented. In addition, the well-known Gilbert and Pollak conjecture on the connection between the length of MST and the length of minimal spanning tree (SMT) is formulated. The investigation of this conjecture and questions connected with it are also presented in [66]. We discuss this conjecture and other similar questions in the last section.

The next attempt in construction of the exact solution of the SPE was undertaken by Cockayne in 1970 [32]. In that paper he improved his previous algorithm with the help of some reduction procedures based on some results from [66]. As a result, an algorithm which can be applied for solution of the SPE with $n \leq 30$ is described. The algorithm reduces first the initial problem to the problems with 6 or less terminals.

Basing on this algorithm and some previous results, Cockayne and Schiller [33] develop a program of solution of the SP which in operation generates the FT's of every subset W of A and constructs the corresponding FST's (if they exist). All but the shortest FST's for W are discarded. The shortest connected union of the remaining FST's which spans A is the SMT for A. Then the minimal FST is found and, on the basis of its union, the construction of the SMT for the given set of terminals is carried out. In this procedure the number of not discarded FST's is too large for the available storage even for a relatively small n, and as a result, many subsets of A are processed more than once.

Extending these investigation, Boyce [16] considers the problem to decide by computation whether there exists the MFST for a given set of terminals, and if so, to find at least one. Boyce introduces some new definitions with appropriate examples to clarify the situation of the full ST's that are not FSMT's (i.e., not-minimal-length connecting networks). The author reorganized the computational algorithm to speed up Cockayne's program. Numerical experiments for $n \leq 10$ were presented. A modification of the algorithm which is able to solve problems with $n = 12$ is presented in [17].

Winter in 1981 announced the program 'GEOSTEINER' for exact solution of the SPE, which can solve problems with $n \leq 15$. 'GEOSTEINER' presented in [157] uses some topological and reduction ideas proposed by Melzak, Gilbert, Pollak, but on the basis of these ideas a complex construction which uses more fine methods and procedures is proposed. As a result, the question about non-existence or non-optimality of the FST for a given FT is solved more efficiently, and the above mentioned problems with storage appear at more high dimensions. For all examined point sets with no more than 15 terminals, the number of discarded FST's never exceeded 100 and was usually considerably less. As a result, the FST's were stored in core, and the formation of their unions was postponed until the FT's of all subsets of A had been processed. The computational experiments are presented for each n, $3 \leq n \leq 15$ and $n = 25$. The point sets were generated with terminals uniformly distributed inside the unit circle. For instance, all examined point sets with not more than 15 terminals were solved in less than 30 seconds (UNIVAC-1100). For $15 \leq n \leq 20$, the SMT's were not obtained within this time. The author assumes that a more efficient procedure for the formation of FST's could improve the overall performance of GEOSTEINER.

Cockayne and Hewgill proposed in [34] a further improvement of this algorithm at the expense of connection to it a reduction procedure at first stage. As a result, about 70% of problems with 30 points were reduced to problems on 17 and less points, which then GEOSTEINER was applied to. The total time of calculations did not exceed 200 sec. This reduction enables the second part to process considerably faster. The machine times depend heavily on the geometry as well as on n. Computational experiments indicate that the increase in problem size from n to $n + 1$, $10 \leq n \leq 17$, causes at least a doubling of average computational times for GEOSTEINER without this improvement, and the value $n = 17$ was the limit for reasonably quick (200 sec.) solutions of randomly generated problems.

Thus, two main directions of investigations, an exhaustive search of the FST and reduction methods, lie on the basis of the listed algorithms. That is why a number of publications is dedicated immediately to the reduction procedures.

At present, only three different main reduction procedures for the SPE exist. The first procedure was proposed by Gilbert and Pollak [66]. Suppose two lines intersect at 120° and cut the plane into two 60° wedges and two 120° wedges. Let R_1 and R_2 denote the two closed 60° wedges and let x denote the point at which R_1 and R_2 meet. Let F_i denote the set of fixed points in R_i, $i = 1, 2$. If $F_1 \cup F_2 = A$, then the SMT on A is the union of the SMT on F_1 and the SMT on F_2 and the shortest edge connecting F_1 and F_2.

The second decomposition theorem, as mentioned above, was presented by Cockayne [32]. Let P_1 denote the convex polygon which is the boundary of the convex hull of A. Let (p, q, r) be a triple of fixed points such that p and q are on P_1, r is either on or within P_1, $\angle pqr \geq 120°$, there are no other terminals within the triangle pqr. Let P_2 denote the polygon (called the Steiner polygon) obtained by deleting the triangle pqr from P_1. We can now substitute P_2 for P_1 and proceed. If no more such triples pqr can be found, we obtain the Steiner hull of P. Suppose that for some P_i the triple (pqr) we find is such that r is also in P_i. Let $f_1, \ldots, f_m, f_1$ denote the ordered sequence of terminals on P_i, where $f_i = p$, $f_{i+1} = q$, $f_j = r$, $i + 1 < j$. Let F_1 denote the set of terminals bounded by the polygon $f_1, \ldots, f_i, f_j, \ldots, f_m, f_1$, and let F_2 denote the set of terminals bounded by the Steiner polygon $f_j, f_{i+1}, \ldots, f_{j-1}, f_j$. Then the SMT on F is the union of the SMT on F_1 and the SMT on F_2.

The third method of decomposition is described in the recent paper [81]. Let $P(x_i, \ldots, x_m)$ denote the polygon whose vertices are $x_i, \ldots, x_m$. Let $P(A)$ denote the Steiner polygon and let a, b, c, d be four points on $P(F)$ such that $P(a, b, c, d)$ is a convex quadrilateral, $\angle a \geq 120°$, $\angle b \geq 120°$; let the two diagonals meet at 0, then $\angle boa \geq \angle a + \angle b \geq 150°$. Then the SMT on A is the union of the SMT on F_1 and the SMT on F_2 and the edge $[a, b]$ where F_1 (F_2) is the set of terminals lying inside the area bounded by $P(F)$ and $[a, b]$ ($[b, c]$) and outside $P(a, b, c, d)$.

This decomposition theorem can be considered as an extension of Cockayne's result consisting in the transition from deleting a triangle to deleting a quadrilateral. The authors think that this result provides a 50% increase in the potentialities of the decomposition theorem. But up to now there is no exact algorithm based on this result.

Generation of the ST is the subject of the paper [79]. The approach proposed in that paper is an extension of Melzak's procedure.

Now we turn to some heuristic algorithms for the SPE which are proposed in [20, 88, 90, 132].

Kelmans in [88] describes an evident local algorithm for generating an approximate solution by sequential addition of terminals to the fragment constructed on the previous stage of solution.

In Korhonen's algorithm [90] a minimal spanning tree is transformed into the ST. Chang [20] develops an iterative procedure which, when applied to an MST, constructs a tree with certain desirable properties. For $n \leq 4$ this algorithm converges to the proper tree. Experimental studies for $n \leq 30$ seem to indicate that it yields good suboptimal solutions to the SPE. The problems with not more than 100 terminals can be handled (IBM 360/65). This algorithm is iterative in nature and can be terminated at any stage.

One of the most efficient heuristics is suggested in [132]. This algorithm constructs an approximate solution in time $O(n \log n)$. It incorporates two phases: a reduction and an expansion. Initially, the set A is triangulated. Within each triangle, the local optimum is found. This triangulation is the Delaunnay triangulation and demands $O(n \log n)$. The second phase is based on the properties of the Voronoi diagram and the minimal spanning tree. During this phase, the solutions for each triangle are reconstructed into the solution for A in time $O(n \log n)$ (constructing the SMT takes $O(n \log n)$ time and the concatenation process takes $O(n)$ time). Computational experiments for $n \leq 50$ are presented. All problems were solved in less than 1.5 sec. The ratio L_{SMT}/L_{MST} for the $O(n \log n)$ algorithm is at least as good as for the previous $O(n^4)$ algorithm. We denote the lengths of minimal Steiner tree and spanning tree by L_{SMT} and L_{MST} respectively. The same approach was used for solution of the problem on the surface of a sphere [169].

When *NP*-completeness of the SPE had been proved, some authors began to pay more attention to the polynomial solvable special cases of this problem. Thus, in [65] Georgacopoulos and Papadimitriou proposed a method of solving the SP with complexity $O(n^2)$ in which only one S-point is allowed. A method of dividing the plane into $O(n^2)$ regions in such a way that, when a new S-point is added to each of this region, the resulting SMT has a fixed topology, formed the basis of this algorithm. The optimal allocation of a new point within the region can be found in a constant time per region.

Provan [116] investigates the SPE with the additional property of convexity, i.e., the terminals lie on the bound of a convex region. The SPE is *NP*-hard, i.e., in the general case there is no fully polynomial approximation scheme (i.e., an algorithm which for any A and any $\varepsilon > 0$ in a polynomial time with respect to n and $1/\varepsilon$ finds a ST such that the ratio of the length of this ST to the length of SMT does not exceed $1 + \varepsilon$) if $P \neq NP$. But in the above mentioned special case Provan presents a fully polynomial approximation scheme for the SPE. This algorithm takes $O(n^6/\varepsilon^4)$ time.

Let R be a polygonally bounded region of the plane, i.e., a connected closed region of the plane, whose boundary is made up of a finite number of straight-line segments, and we have to find the SMT for A in R, i.e., a spanning graph for A in R having the minimal length. An algorithm of the solution of the SPE for terminals within such a region is constructed in [117]. That paper extends the previous result [116] to the SMT with restrictions.

Three simple special classes of terminals are considered in [43, 45, 88]. Here analytical formulae for the length of SMT are presented.

In [45] the set of points lies on a circle with at most one large distance between two consecutive points. In [43] the terminals are the points of intersection on a regular zig-zag lines, i.e., a connected sequence of line segments which turn in alternate directions with a constant angle.

As it has been said in Introduction, a number of papers are dedicated to the problems whose titles have the term Steiner problem. For example, the following publications can be quoted.

Trietsch [142, 143] considers two problems, which he calls generalizations of the SPE. In the first problem [143] it is required to interconnect N networks on the plane by the set of edges of the minimum total length. The edges are straight segments, and it is possible to make connections with the vertices of the networks or with any points of the edges. The use of S-points is also allowed. The author proposes a finite algorithm for this problem similar to those used for the SPE [33]. This problem can be generalized in order to include flow-dependence costs for various links (see Section 6). In the case of single nodes as degenerate networks this problem is reduced to the SPE. If exactly one of these networks is non-degenerate, we obtain the Steiner network augmenting problem [142]. In that paper the author proved that the problem can be solved in finite time with the help of the technique used for the SPE.

The analogue of SPE in which terminals are simple polygons is considered in [30]. In [104] the case of two-criterion SP is studied. The surveys of applications of the SP to technology, circuit design, etc. are presented in [92, 102, 104].

4. THE STEINER PROBLEM IN THE PLANE WITH RECTILINEAR METRIC

If in the previous statement of the problem the rectilinear metric substitutes for the Euclidean one, the rectilinear Steiner problem (RSP) is obtained. In this case the ST covers terminals using only vertical and horizontal lines. Here the distance between two points with coordinates (x_i, y_i), (x_j, y_j) is equal to $|x_i - x_j| + |y_i - y_j|$. S-points are also allowed. The problem is to find the rectilinear Steiner tree with minimal length (RMST).

In particular, the papers [70, 73, 74, 92, 100] are dedicated to numerous applications of this problem.

In the next section the SP's in networks are discussed. The RST was originally formulated by Hanan [71]. In that publication Hanan proves that the RSP is a special case of the SPN. More exactly, let two sets of real numbers $X = \{x_0, \dots, x_{n-1}\}$ and $Y = \{y_0, \dots, y_{n-1}\}$ be given. The points of the plane whose first coordinates belong to X and the second coordinates belong to Y are the vertices of the grid graph $G = (V, E)$, and two vertices $(x_i, y_j), (x_k, y_l)$ are connected by an edge if and only if $|i-k| + |j-l| = 1$.

The ordered sets of the first and second coordinates of the terminal vertices form sets X and Y, the grid graph on the terminal vertices is denoted by $G(A)$. Hanan proves that the SMT in RSP is a subgraph of $G(A)$. Thus, all results which are related to the Steiner problem in networks are applied to the problem under consideration. But due to historical reasons and to the features connected to applications, the given problem is traditionally investigated separately. This problem, as Garey and Johnson [63] showed, is NP-complete. The publications devoted to this problem consider either heuristic methods or polynomially solvable special cases. And exact methods of its solution which are of no applied interest are studied in the general range of the SPN.

A brief survey of heuristics is presented in the introduction to Richards's paper [120] and in Winter's paper [161], who considers the given problem as a special case of the SPN.

Some special properties of the RST are studied in [25, 54, 63, 71, 72, 73, 75, 76, 106]. Exact methods of solution are proposed in [71, 166]. For the case of arbitrary n there exists only one exact algorithm devoted specially to RSP, proposed by Yang and Wing [166]. It is based on the branch-and-bound method and proves to be applicable only for $n \leq 10$. In that paper a suboptimal branch-and-bound algorithm is proposed, which proved to be applicable for $n \leq 30$. The comparison with the known exact algorithm shows a remarkable closeness of the characteristics of this algorithm to the optimal values [167]. It simply uses Prim's algorithm [114] but instead of choosing one of the two possible one-bend orientations of a new wire it explores both.

Hwang's result [75] (if for a set of terminals A there is no SMT, with terminal degrees more than one, then there exists a solution which has one of two fixed topologies described in the paper), which characterizes the SMT for one class of RSP, can be used in construction of algorithms to solve the SP. A simple proof of this theorem is given in the recent paper by Richards and Salove [174]. In the above mentioned paper [71] Hanan formulated and proved for this problem properties (1)–(5) which were similar to the properties pointed out for the case of SPE (see Section 3). In addition he studied in detail exact methods for $n = 3, 4, 5$. A heuristic algorithm with complexity $O(n^2)$ is proposed. This problem is discussed in the paper [73] too.

This heuristic which was the first heuristic for the given problem is a generalization of Prim's algorithm. The terminals are numbered in increasing order of the first coordinate. The heuristic begins with the vertices A_1 and A_2 and connects A_1 and A_2 by the shortest paths in $G(A)$. From the set of such paths it selects one having the largest portion coinciding with the line $x = x_2$. If the partial tree T_k for $A_1, \ldots, A_k$ has been constructed, A_{k+1} is added to T_k by the shortest path having the largest portion coinciding with the line $x = x_{k+1}$. If T spans A, the heuristic terminates.

We give a detailed description of this method because its modification proposed in [120] is the basis of the heuristic for RSP which is the best at the moment. The improvement of the algorithm takes place on the stage of the connection of a terminal and the constructed fragment due to looking through a number of additional variants. For each new point, the set of $O(n)$ terminals, S-points, corners and wires of the current tree must be inspected and the shortest wire is determined in linear time. This leads to an $O(n^2)$ runtime. It is shown that this method has the $O(n \log n)$ runtime using computational geometry methods. But in the case of sequential searching it takes $O(n^2)$ time in the worst case. However, it is shown that this approach runs in $O(n^{3/2})$ expected time, for n points randomly selected from a $p \times p$ grid. Empirical results are presented for the problems up to 10000 points. A comparison with another heuristics shows pronounced superiority of this method (see Table 1).

Fu [62] developed the heuristic transforming the SMT by deleting on each step the path in the spanning tree which contained in its interior only S-points of degree 2 (if any), and by reconnecting the resulting components by another path in a special manner. Fu claimed that this heuristic produced an optimal solution, but this assertion was disproved by Hanan [72].

Smith and Liebman [129] suggested an $O(n^4)$ heuristic. It begins with selecting a linear-sized subset of n^2 vertices as candidates for S-points, then these vertices S_i, $i = 1, \ldots, n^2$, are sorted in the non-decreasing order of costs of the SMT for $A \cup \{S_i\}$

and, as a result, in some cases a current SMT is replaced by the SMT for $A \cup \{S_i\}$. The authors also suggest three different ways of selection the subset of S-points which seem to be comparable to one another with respect to both average execution time and costs of obtained suboptimal solutions.

In [130] an $O(n \log n)$-time approach is proposed, which is based on an iterative improvement of the rectilinear SMT found with the help of the Delaunnay triangulation.

In [101] the so-called single net wiring problem as a SP in graphs with the rectilinear metric is formulated. An algorithm to find a suboptimal solution satisfying only the minimum length constraint is described and the obtained results are compared with the existing algorithms given in [114, 167], adapted for the rectilinear case. This algorithm uses a three point connection scheme instead of joining the nearest unused vertex and runs in $O(n^2)$ time. The size of nets for computational experiments ranges from 5 to 35 points.

Hwang [77] proposed an $O(n \log n)$ modification of the heuristic from [101]. This improvement of the runtime due to the following result.

Shamos and Hoey in 1975 used some properties of the Euclidean distance to develop an $O(n \log n)$ algorithm for the construction of the Voronoi diagram for A based on the divide-and-conquer method and then derive the minimal spanning tree from the Voronoi diagram in less than $O(n \log n)$ time. While deriving a SMT from the Voronoi diagram is valid for an arbitrary distance, the algorithm for the Voronoi diagrams depends critically on the distance function. Hwang [78] gives an $O(n \log n)$ algorithm for the Voronoi diagrams for the rectilinear distance and obtains an $O(n \log n)$ algorithm for the rectilinear minimal spanning trees. He uses standard divide-and-conquer techniques. In the algorithm from [101] an auxiliary procedure for the SMT in $O(n^2)$ time is used, but Hwang uses $O(n \log n)$ procedure instead of this one.

Bern and de Carvalho [8] investigated Kruskal's based approaches [96] suggested by Thompson. Thompson's algorithm is executed in $O(pn^2 \log n)$ time and its variations take $O(pn^2)$ time.

It is accepted for the comparison of algorithms to use the ratio (in percent) $(L'-L)/L$, where L is the length of MST, L' is the length of ST which is a result of the heuristic. This ratio is called the percentage.

In Table 1 the results of comparison of seven heuristics are presented [120].

Servit's paper [126] is of applied interest. Servit compared the performance of several simple heuristic algorithms in a large number of examples of real problems of printed circuit boards. Such experiment is considerably more natural and informative than

Table 1.

Ref.	Investigators	Year	Time, sec.	Problem size	Percentage
[166]	Yang, Wing	1972	185	35	11
[129]	Smith, Liebman	1979	34	40	7
[130]	Smith, Lee, Liebman	1980	1	40	8
[101]	Lee, Bose, Hwang	1976	5	35	9
[77]	Hwang	1979	–	–	9
[8]	Bern, Cavalho	1985	1	40	9
[120]	Hanan, Richards	1989	56	10000	4

examples with random sets of points. In this case the computational experiment may give quite different results in comparison with the uniformly distributed instances. Let us give a brief resume of this investigation. The following eight heuristics are compared. STAN algorithm is the straightforward generalization of the Prim algorithm of complexity $O(n^2)$. HAN1 and HAN2 algorithms are suggested by Hanan [79] and have the runtime $O(n^2)$. SRV1 and SRV2 are straightforward simplifications of HAN1 and HAN2 algorithms. The simplification consists in the fact that the shortest path is not constructed between a next vertex and the current tree but between this vertex and the path constructed in the previous step. Since the ordering can be implemented in $O(n \log n)$ time and the remaining steps, in $O(n)$ time, SRV1 and SRV2 are $O(n \log n)$ algorithms. RECT is based on the computation of the semiperimeter of the smallest rectangle enclosing A. Some results related to this method are obtained by Chung and Hwang [26] (see Section 6). AVBR is based on the estimation of the length of MRST and has the complexity $O(n^2)$.

The analysis of experimental results leads to the conclusion that the AVBR algorithm provides, in all the parameters investigated, worse results than the SRV1 algorithm. Similarly, by virtue of the comparison with the HAN2 algorithm, the PRIM and STAN algorithms can be excluded. The SRV2 algorithm can be excluded by comparison with the HAN1 algorithm. With regard to computational time, the algorithms can be ordered as follows: RECT, SRV1, HAN1, HAN2. Quite an opposite order is obtained when the quality of the results is considered.

A survey of recent publications and algorithms for the RST problem concerning the applications of these algorithms to the problems in circuit design are given in [92].

Two probabilistic partitioning algorithms for RST problem are obtained in [89]. The algorithms subdivide n given points, uniformly distributed on the unit square $[0,1]\times[0,1]$, into small groups, construct the minimal RST for each small group and then patch the subtrees together to form a near-optimum RST. Let T_0 be a minimum RST and T_1, T_2 be rectilinear spanning trees constructed by the algorithms. Based on the probabilistic approach introduced by Karp for the Euclidean travelling salesman problem, for any given integer $t > 0$ the first algorithm runs in $O(f(t)n + n \log n)$ time and produces T_1 such that $L_{T_1}/L_{T_0} \leq 1 + O(1/\sqrt{t})$ with probability approaching 1 as $n \to \infty$ while the second algorithm has an expected running time $O(g(t)n)$ and produces T_2 such that $L_{T_2}/L_{T_0} \leq 1 + O(1/t)$ with probability approaching 1 as $n \to \infty$.

Now we proceed to polynomially solvable special cases of the RSP.

In the above mentioned paper [116] Provan gives a definition of the rectilinear convexity, which means that the terminals lie on the boundary of a rectilinear convex region. For this problem an exact algorithm of complexity $O(n^6)$ is constructed. The same result is obtained for the SPN too.

The RSP for the special constructions (ladders) is considered by Chung and Graham in [23].

A subnetwork $G(A)$ is called a rectangular tree (RT) if it can be obtained using the following recursive construction. The right angle (a connected network whose edges lie on the sides of the right angle) is a RT. Another right angle can be added to the available RT by identification of an edge of this angle with that one of the edges of RT which lies on the outer face in such a way that vertices of degree 4 are not originated. The minimum distance RT is such a RT that the distances between any pair of vertices in RT itself and in $G(A)$ are equal. The description of all classes of such RT's is obtained in [54]. The main result of the paper [54] is the proof that the existence of the minimum distance RT spanning all terminals involves the existence of SMT with the same length. Since RT(A) is an outerplanar graph, the linear time algorithm can be applied. The authors completely enumerated all classes of the minimum distance RT's. Note that Winter [161] was the first who had announced this result.

In [2] two special cases in which the elements of A all occur either on parallel lines or on the boundary of a rectangle are considered. For the first case a linear algorithm and for the second one a cube algorithm were obtained.

A linear algorithm for the second case was presented in [35]. The authors prove that the RMST in this case necessarily has one of the ten fixed topologies, and the consideration of the SMT in each of these topologies is carried out in linear time. The same result was obtained by Agarwal and Shing [1]. Their algorithm is similar to that one from [35].

Trubin [141] investigates a subclass of SP in the plane with a special metric (rectangular metric) and suggests a polynomial algorithm for this subclass.

5. STEINER PROBLEM IN NETWORKS

The statement of the problem has already been presented in Introduction. In this case for undirected network $G = (V, E, c)$ with p vertices, m edges, a cost function c: $E \to R$ and $A \subseteq V$ ($|V| = p = n + s$, $|A| = n$, $|S| = s$, $|E| = m$), we have to find a subnetwork G_A which spans all vertices of A and the sum of its edge costs takes the minimum value. This subgraph is called the minimal Steiner tree (SMT). The vertices belonging to $V \setminus A$ are called S-points (S-vertices, Steiner vertices)

The main attention in this section will be paid to the results obtained after 1985. However briefly we will try to give a related picture of results available in this field. The full picture containing the detailed description of algorithms can be obtained from Winter's survey [161], Voss's monograph [146] and Maculan's survey [105].

Hakimi [69] was the first who had formulated the SPN and presented a topology enumeration algorithm. Hakimi's approach is similar to Melzak's one for the SPE. In Hakimi's approach one calculates the minimal spanning tree for each of the possible subsets of points starting with A and ending with V. To calculate the minimal trees, the method of Kevin and Whitney is used. The algorithm runs in $O(n^2 2^k + (n + k)^3)$ time. Hakimi states that his algorithm requires $O(n^4)$ if $n - k \leq 2\log n$.

Levin's method [102] uses the general combinatorial optimization approach of dynamic programming and is based on the computation and storage of the total weight of the minimal spanning tree for each possible subset of points. These weights are calculated iteratively, the weights at the kth stage being derived from the information gained at the $(k - 1)$th stage. It takes $O(3^n(n + k) + 2^n(n + k)\log(n + k) + m))$ time.

The algorithms constructed by Dreyfus and Wagner [41], Bern [13], Lawler [99], Ericson et al. [53] are also based on the dynamic programming. The method of Dreyfus and Wagner successively decomposes the problem into progressively smaller and smaller subproblems until each final subproblem can be solved by forming the matrix of shortest paths between all pairs of vertices in G. Only the optimal solutions for relevant subsets of points are constructed. Each optimal solution is stored and may be used in the calculation later. This algorithm runs in $O((n + k)3^k + (n + k)^2 2^k + (n + k)^3)$ time. The Lawler's algorithm takes $O(2^n k^2 + (n + k)^2 \log(n + k) + m)$ time. In the paper [53] a similar dynamic programming algorithm for a more general problem of finding the minimal cost flow in a network with additional restrictions is proposed. Application of this algorithm to a case of SP in a planar network in which all terminals lie on a single face leads to a polynomial-time algorithm with complexity $O(nk^3 + (n \log n)k^2)$. And if the number of faces, on which the terminals lie, is equal to f, then the complexity is $O(nk^{3f} + (n \log n)k^{2f})$. It should be noted that Provan [116] independently discovered the polynomial algorithm for the case $f = 1$.

Bern [13] improved the result of the paper [53] and obtained an algorithm with complexity $O(nk^{2f+1} + (n \log n)k^{2f})$. He also gives other bound of the complexity of the algorithm from [53] which is exponential only in the number of required nodes that do not lie on a common facial boundary. The SP for the planar network embedded in the plane such that w terminals lie on the boundary of the infinite face can be solved in time $O(nw^2 3^{k-w} + (n \log n)w^2 2^{k-w})$.

In particular, from this result a polynomial $O(w^3 k^2 3^{k-w})$ algorithm for the RSP follows. This is an improvement of Provan's result [116].

The third approach to the exact solution of SPN was presented by Aneja [3]. He formulated a SPN as a set covering problem and used a modification of the cutting plane algorithm for the general set covering problem proposed by Bellmor and Ratliff and the dual simplex algorithm in the relaxed linear program associated with the integer programming problem. He also used some heuristic rules for improving the computational efficiency of the algorithm.

Foulds and Gibbons [57] presented a tree search procedure based on a bound derived by finding for each terminal the minimum cost of connecting that vertex to some other vertex. Computational experiment indicated that only relatively small problems can be solved.

The method suggested in [128] also employs the integer programming technique of branch and bound enumeration. The procedure systematically examines a series of partial solutions and bounds used to discard the partial solutions which cannot be a part of the minimal ST.

Some of the above mentioned algorithms are compared in [58]. The algorithms were coded in ALGOL and ran on Burroughs B6700. Results of the computational experiments are natural with the point of view related to their complexity. The algorithm of Dreyfus–Wagner dominates Levin's algorithm. The other results are given in Table 2, where * denotes that the time of calculations exceeded 100 sec.

Table 2.

		Average times (sec.)			Range (sec.)		
n	k	Dreyfus, Wagner	Hakimi	Shore	Dreyfus, Wagner	Hakimi	Shore
10	3	< 1	5	3			1–4
10	5	9	1	2	9–10		1–3
10	8	*	< 1	1			1–2
20	5	36	*	25	35–36		16–36
20	10	*	88	21		82–95	12–29
20	15	*	2	22			9–51
30	5	45	*	*	45–46		72–100
30	15	*	*	90			56–100
30	25	*	6	80		6–7	53–48

Note that the running time of Aneja's algorithm for the problems with $n \leq 50$, $k \leq 20$, $m \leq 60$ was in the range of 40 sec. But, for example, he could obtain an exact solution only of four problem among ten attempts for $n = 50$.

Yang and Wing [165] develop an algorithm which is similar to Shore's algorithm but is less efficient.

Beasley [6, 7] proposed three algorithms for the exact solution of SPN, which was considered as a problem of a zero–one integer programming and was solved by using the Lagrangian relaxation. The algorithm for the shortest path problem and subgradient optimization were used in an attempt to maximize the lower bounds obtained from the Lagrangian relaxations of the problem. In [6] computational experiment with $n \leq 50$, $m \leq 200$, $k \leq 50$ are presented. In [7] the SPN is formulated as a shortest spanning tree problem with additional constraints. By relaxing these additional constraints in the Lagrangian fashion the author obtains a lower bound which can be used in a tree search procedure for the problem. Problem reduction tests derived from both the original problem and the Lagrangian relaxation are given. This algorithm solves problems involving the connection of up to 1250 vertices in a graph with 62500 edges and 2500 vertices. Numerical experiments were carried out with FORTRAN program on Cray X-MP/42.

The same approach is considered in [115, 163], but for the calculation of the lower bounds another procedures are used. Wong in [163] investigated the SP in directed graphs. In this problem the minimal directed subtree that connects a root node to every terminal node is required. The author gives a procedure for obtaining lower bounds for the minimal ST. Computational results indicate that the method is effective in solving problems containing up to 60 nodes and 240 arcs.

Liu [103] also considered a lower bound for the SP in directed graphs. He presented a new integer programming formulation. Wong's algorithm is used as a subroutine. Also some simple heuristics are used to obtain upper bounds. For every problem with fixed number of nodes and terminals, 10 asymmetric directed graphs are randomly generated, the average running time in the case of $p = 100$, $n = 40$ is 200 sec.

In [110] the Steiner problem for directed graphs without directed cycles but with a node from which a directed chain emanates to every other node is considered. An algorithm to solve SP in such graphs is given. It is an algorithm to solve a linear zero–one programming problem. The algorithm was realized on CDC-6600, the results are given for $p \leq 320$, $n \leq 100$.

Whereas in the previous approaches the main attention was paid to obtaining lower bounds, particularly to the method of Lagrangian relaxation, the other direction, well developed at present, concerns with the development of reduction procedures. These algorithms are more simple in realization and allow us to solve the problems of sufficiently large size. The papers [5, 50, 51, 82] concern this approach.

In [82] a reduction procedure based on a heuristically derived upper bound of the length of SMT and its extension to the SP with degree-dependent costs are presented.

Balakrishnan and Patel [5] give some characteristics of the optimal solutions of SPN and propose a reduction procedure based on these properties. Also they derive an estimate of the expected reduction achieved by this method under a set of probabilistic assumptions and demonstrate that their scheme produces asymptotically optimal reduction. All programs are coded in FORTRAN and the SMT is calculated using Kruskal's algorithm [96]. The results are given for $n \leq 20$, $k \leq 30$, $m \leq 60$. They also propose a tree generation algorithm that exploits the special structure of the SPN to construct its constrained minimal spanning tree.

The papers [50, 51] improve the existing tests for the reduction of the SPN size by eliminating vertices from the graph and develop new techniques based on the bottleneck approach. The latter include optimal edge detection, edge elimination and node elimination. Computational experiments are considered in detail. All programs were coded in PASCAL. The problem size is ranged to $p \leq 200$, $k \leq 100$, $m \leq 4950$. The edge densities vary from sparse to complete with uniformly distributed random Euclidean weights. The algorithm solves the majority of the test problems by reduction tests only, and reduces the size of the remaining problems to at most one fourth of the initial size.

In [124] an original analogue method for the SPN based on the thermodynamic simulation is proposed.

Now we proceed to the heuristic algorithms. It should be noticed that some of the above mentioned exact algorithms, if they are interrupted on some step, can be used as a heuristic. For example, Aneja [3] and Wong [163] constructed heuristics on the basis of their algorithms. These heuristics are called a set covering heuristic and a dual ascent heuristic.

The set of the existing at present heuristics can be divided into the following classes.

The heuristics on the basis of the shortest path problem. The first shortest path heuristic was developed by Takahashi and Matzuyama [138]. It computes a ST by combination of the processes of finding the shortest paths and finding the SMT. The procedure of finding the SMT is analogous to the procedure presented by Prim [114] for graphs without S-points. The algorithm runs in time $O(kp^2)$. As shown in [138], the worst case ratio of this heuristic is $\beta = L'/L = 2(1 - 1/k)$, and this estimation is attainable. Shaohan [127] suggested a modification of this heuristic.

The heuristics on the basis of the distance network. A distance network heuristic was given independently in [52, 93, 112]. This heuristic is a generalized version of the SMT for A. The algorithm first finds all the shortest paths between vertices in A, and then computes the SMT for A based on the distances of these paths. This algorithm has the worst case time complexity $O(kp^2)$ and the worst case ratio of this heuristic is $\beta = L'/L = 2(1 - 1/l)$, where l is the number of leaves in the SMT.

Plesnik [112] and Sullivan [136] propose a modification of this method with the help of a preliminary increase of the subgraph for the shortest path procedure by adding q S-points. It takes $O((p-k)^q k^2 + p^3)$ time, and the worst case ratio tends to $2-q/(k-2)$.

In [164] the approach of [93] is used, with Kruskal's algorithm instead of Prim's algorithm and another combination of the both stages. The complexity of this algorithm is $O(m \log p)$. This algorithm is especially effective for sparse graphs.

Widmayer [156] suggested a modification of this method; this algorithm computes an approximate solution of SPN in time $O(m + (p + \min(m, k^2)) \log p)$, with the worst case ratio no more than $2(1 - 1/l)$.

A further improvement of this algorithm was proposed by Mehlhorn [107], where the complexity fell till $O(p \log p + m)$. The worst case ratio in this method is no more than $2(1 - 1/l)$ too.

In [97] the SPN is investigated from the probabilistic point of view. Assuming the edge probability for a random graph, they investigated the problem in this setting and presented a lower bound for the optimal solution that holds for almost all graphs. Two different polynomial algorithms, one of which was proposed in [92], are used for the calculation of this bound. The authors show that the solution achieved by these algorithms are very close to the lower bound.

The heuristic of Kou and Makki [94] also belongs to this class and is a further development of the previous algorithms. Its worst case ratio is the same as in the previous cases, and the complexity is

$$O(m + k\log k + t\log\alpha(t,k)),$$

where $t = \min(m, k(k-1)/2)$, $\alpha(t,k) = \min\{i\colon \log^{(i)} t \le t/k\}$.

The average distance heuristics. The first average distance heuristic was suggested by Rayward-Smith [118]. For the cases $k = p$ and $k = 2$ this algorithm produces an exact solution. At each iteration, the heuristic examines the list of trees which will be subtrees of the final tree. Initially the list consists of isolated terminals. Using a special distance function for calculating distances between nodes and trees, two subtrees of the list are selected and joined by the minimum cost path in G. Hence, after $k-1$ iteration, the list contains just one tree spanning all terminals. The complexity is $O(p^3)$. Experimental results show that this method performs satisfactorily in the rectilinear case.

Testing this algorithm, Waxman [152] proves that in the worst case its solution time is twice as great as the optimal one and this bound is attainable.

Multiple source shortest path heuristic. The Wang's algorithm belongs to this class of heuristics [151] and computes a ST by linking closest subtrees of G. It starts with every vertex in A being a tree and ends when all vertices in A belong to the same tree. The runtime is $O(kp^2)$ in the worst case, and it is $O(p^2\log k)$ in the best case and on the average, under suitable assumptions. The quality of this algorithm is bounded as that of the algorithm from [93].

Rayward-Smith and Clare [119] carried out a detailed comparison of the three classes of heuristics. They took the algorithms of the papers [93, 118, 138]. The results are quite natural, for example, the first one gives the best result in quality and runs in the longest time.

Polynomially solvable cases of SP are investigated in [4, 10, 19, 37, 38, 115, 116, 121, 147, 148, 154, 155, 160].

The case of the series–parallel networks was discussed in Section 2. The linear algorithms for SPN in this case are described in [115, 148].

Outerplanar networks are investigated by Wald and Colbourn [147]. A graph is outerplanar if it can be embedded in a planar graph such that all its vertices are on the exterior face. Every outerplanar network is a series–parallel network, therefore the linear algorithm for this case is a special case of the previous one.

The Halin network is obtained by a planar embedding of a tree and then by adding the edges between the consecutive leaves to form a cycle. Winter [160] constructed the linear algorithm for the SPN in the Halin networks.

The polynomial algorithms proposed by Bern [10, 13] and Provan [116] for networks with all terminals belonging to a fixed number of faces have been mentioned above. Provan [173] paid attention to the constructions (called Steiner hulls) joining algorithms to solve SPE and SPN.

A network, whose every cycle with four or more edges has a chord and every even cycle with six or more edges has a chord dividing the cycle into two paths, each containing an odd number of edges, is called a strongly chordal graph. The algorithm for SP of complexity $O(n^3)$ for this class of networks with the unit cost edges was developed in [154, 155]. The same restrictions on the weights are considered in [37], but an additional homogeneity restriction is imposed on the structure of the network. A set of two or more nodes such that any remaining node in G is adjacent either to all or to none of its nodes, is called a homogeneous set of nodes. The main result of [37] states that an instance of the SPN on G is polynomially reducible to an instance of the same problem on an induced subgraph of G whenever G contains a homogeneous set of nodes. This fact allows us to solve in polynomial time the SPN on a class of graphs defined in terms of homogeneous sets. A polynomial algorithm for the recognition such graphs is also given.

A graph is distance-hereditary if each cycle of five or more vertices has at least two crossing chords (where (u, v) and (w, z) cross if the vertices u, v, w, z are distinct and placed in this order on the cycle). D'Atri and Moscarini [38] proposed a polynomial algorithm for the SPN in such graphs with unit-cost edges. This algorithm takes $O(mp)$ time.

Bern and Plassman [12] constructed an algorithm with $\beta = 4/3$ for the SPN in the case of complete network with edge lengths equal to 1 or 2.

Some generalizations of the SPN are studied in [49, 95, 125, 142, 143, 145, 153, 158, 159]. But there we have the same situation as in the case of SPE and we do not discuss it in detail, but present only the references and some examples.

The SPN in the digraph is a natural generalization of the SPN. This problem was considered above. A heuristic algorithm for this problem using the ideas from the papers [5, 6, 7] is proposed in [172]. Besides, the k-best spanning trees are constructed for obtaining an upper bound. Numerical experiment was carried out for $n \leq 300$.

Krarup [95] formulated the generalized SPN as follows. For an undirected graph G and a set of terminals A an $n \times n$ matrix $R = (r_{ij})$ of required local connections between terminals is given. We find a sub-network G_A of G such that every pair of terminals is locally r_{ij}-connected and the total cost of G_A is minimal. Obviously, this problem is NP-complete. Winter developed linear time algorithms to determine 2-connected and 2-edge-connected G_A when G was an outerplanar [158] or series–parallel [159] graph. He also announced a similar result for the Halin networks [161].

The node-weighted SPN, in which the weights are assigned both to nodes and to edges, and the weight of ST is equal to the sum of the weights of its nodes and edges, is one more generalization.

Segev [125] deals with a special case of this problem, where $n = 1$ and weights of all S-points are negative. This problem remains NP-complete. Lower bounds of the optimal solution and heuristic procedures are proposed.

Duin and Volgenant [49] develop an algorithm for the node-weighted SPN based on reduction tests. They also give a transformation showing that this problem is a special case of the SP in directed graphs, and present a new generalization, the Steiner forest problem (in which for given s we find the subgraph of the minimum total weight that consists of at most m components each containing at least one terminal). The same authors in [170] considered the SP with two weights on each edge and proposed two heuristic algorithms using reduction procedures. The complexity of the both algorithms is $O(kn^2)$.

6. THE GILBERT-POLLAK CONJECTURE AND SOME PROPERTIES OF SP

In [66] Gilbert and Pollak formulated a conjecture on the ratio between the length of the minimal spanning tree and the length of the minimal Steiner tree. Let $L_S(A)$ be the length of the minimal Steiner tree on a set of terminals $A_1, \ldots, A_n$ and $L_M(A)$ be the length of the minimal spanning tree for these points. Gilbert and Pollak conjectured that for any set of n points on the Euclidean plane,

$$L_S(A)/L_M(A) \geq \sqrt{3}/2.$$

The Steiner ratio is defined to be

$$\rho_n = \inf\{L_S(A)/L_m(A)\colon\ A, |A| = n\},$$

where the infimum is over all allocations of n points on the plane.

In the same paper the conjecture was proved for the case $n = 3$. Besides this, it was proved, using Moor's result, that $\rho_n \geq 1/2$ and $L_S(A)/L_M(A) = \sqrt{3}/2$ in the case where the terminals were the vertices of the equilateral triangle.

Pollak [113] investigated the properties of SMT for the case $n = 4$. Pollak's approach is to consider all the possible patterns of minimal trees on n points (there are five pattern for $n = 4$) and to give a separate proof of the conjecture for each distinct pattern. A different approach and a very short proof for the case $n = 4$ is given in [44].

In [46] the authors with the help of this approach proved the conjecture for $n = 5$.

Rubinstein and Thomas [175] reformulated the conjecture as a variational problem in which the vector of $2n$ coordinates of the terminals in R^{2n} was disturbed. This approach was the basis of an interesting and original proof of the conjecture for $n = 3, 4, 5$. In [176] using the same technique the authors proved the conjecture for $n = 6$.

Kallmann [86] proves that the length of any ST with only one S-point is not less than $(\sqrt{3}/2)L_M(A)$ for any n.

The estimation of $L_S(A)/L_M(A) \geq 1/2$ for an arbitrary metric space was obtained in [66]. Graham and Hwang [68] proved that $\rho_n \geq 0.5771\ldots$ for any set of n points in the d-dimensional Euclidean space.

For arbitrary n in the Euclidean plane Chung and Hwang [24] obtained the relation $\rho_n \geq 0.74300\ldots$, and Du and Hwang in [42] proved that $\rho_n > 0.8$. Chung and Graham [27] improved this estimation and showed that $\rho_n \geq 0.82416$.

And finally in [24] Du and Hwang proved the conjecture. They used some ideas from the above mentioned papers of Rubinstein and Thomas. They considered a family of a finite number of continuous concave functions $\{g_i(x)\}$, $i \in I$, on a polytope X and the problem of minimization of the function $f(x) = \max_{i \in I} g_i(x)$ on X. The authors showed that the minimum value of $f(x)^i$ was attained at a finite number of special points in X. As an application, they proved the long-standing conjecture.

For the spaces of an arbitrary dimension d with Euclidean metric, ρ_n can be less than $\sqrt{3}/2$. This problem was considered in [66], where the Steiner ratio is studied in the case where $n = d + 1$ terminals are the vertices of a regular simplex. The conjecture (unproved even for $d = 2$) that just on this configuration the Steiner ratio is attained was stated. The Steiner ratio and the SMT of a d-dimensional regular simplex are not known. In [66] short trees are constructed for simplexes of several dimensions d. Except the cases $d = 3, 4, 5$, these trees are not even the Steiner trees; however, the authors prove a bound on $\rho_n(A)$ close to $(1 + \sqrt{3})/4 = 0.68301$ in the limit as $d \to \infty$. In [22] trees are constructed for regular simplexes to obtain a bound for $\min L_S(A)/L_M(A)$ which comes arbitrarily close to $(3/2)^{1/2}(2^{3/2} - 1)^{-1} = 0.66984$ for sufficiently large d. For dimensions $d \geq 5$ these trees are proved to be SMT.

For the RSP Hwang [75] formulated an analogue of the Gilbert–Pollak's conjecture in the form $\rho_n = 2/3$ and proved that $\rho_n \geq 2/3$ and this bound can be attained for infinite number of n. While proving the bound, the author also gave many lemmas which characterized a SMT with rectilinear distance. Some similar characteristics were investigated in [25, 26, 71].

Three results concerning the mean value $E(L_S(A)/L_M(A))$ for the case $n = 3$ and the Euclidean metric are given in [84]. For the uniformly distribution one obtains $E(L_S/L_M) \geq 0.98$, for the Gaussian distribution one obtains $E(L_S/L_M) \geq 0.96$.

A generalization of the Gilbert–Pollak conjecture is considered in [47, 140]. Gilbert formulated the so-called minimal Gilbert network problem (MGN) as a generalization of the STP by adding flow dependent weights to the arcs.

The purpose of [140] is to generalize the Gilbert–Pollak conjecture to the ratio between the MGN and the regular network (MRN), where extra nodes are not allowed. The authors give the proof that when the regular minimal network connects three nodes by adding exactly one extra point, the ratio is equal to $(2-\sqrt{3})/2$ and that this maximal improvement can be attained only for symmetric case, namely, where the three nodes are the vertices of an equilateral triangle and the weights of the three arcs are equal. Let

$$\rho_n = \inf\{L_{\mathrm{MGN}}/L_{\mathrm{MRN}}\colon\ |A| = n\}.$$

Trietsch and Handler conjectured that $\rho_n = \sqrt{3}/2$ and proved it for $n = 3$. But in [47] Du and Hwang give a counterexample to this conjecture for $n = 4$. In this paper a simple geometric proof for $n = 3$ is given.

Let $L_R(A)$ be the semiperimeter of the smallest rectangle with vertical and horizontal lines which contains A. In the case of the rectilinear metric it is easy to see that $L_S(A) \geq L_R(A)$ for $n \geq 3$, and $L_R(A)$ can be used as a lower bound for $L_S(A)$. Therefore in [25, 71] the value

$$r_n = \max\{L_S/L_R\colon\ |A| = n\}$$

is investigated. The values $r_3 = 1$, $r_4 = 3$, $r_5 = 3/2$ are given in [73]. It is shown in [25] that r_n increases monotonically and tends to $(\sqrt{n} + 1)/2$ and $r_6 = 5/3$, $r_7 = 7/4$, $r_8 = 11/6$, $r_9 = 2$, $r_{10} = 2$.

Chung and Graham in [26] investigated the following question. What is the greatest length $s(n)$ of SMT for a set of n points contained in a unit square? By considering subsets of the regular hexagonal lattice placed in a unit square, it is easy to show that $s(n) \geq (3/4)^{1/4}n^{1/2} + O(1)$, and Few [26] proved that $s(n) \leq n^{1/2} + 7/4$. In [26] the estimate $s(n) \leq 0.995n^{1/2}$ was proved for sufficiently large n. For the rectilinear metric the maximum length $s(n)$ satisfies the inequalities $n^{1/2} + O(1) \leq s(n) \leq n^{1/2} + 1 + O(1)$. In fact, $s(r^2) = r + 1$ if $n = r^2$. The conjecture that $s(n) \leq n^{1/2} + 1$ for all n was stated.

In Bern's paper [9] the RSP is investigated. Two probabilistic results are obtained. Given n points distributed uniformly on the unit square, the length of the shortest spanning tree, the rectilinear ST, the travelling salesman tour, or some other functional on these n points with probability tending to one is asymptotically equal to $\beta\sqrt{n}$ for some constant β (different for different functions) as $n \to \infty$. Bern proves that the constants in the cases of rectilinear spanning tree and RST are really different. The expected value of the minimum number of Steiner points in the shortest RST grows linearly with n and is not less than $0.039n$.

Jain [89] applied a probabilistic approach to investigate the relative tightness of linear programming relaxation bound for the integer programming for emulation of the SPN. Under two different random models of a network he shows that the aggregate linear programming relaxation provides a rapidly converging bound for the MST and characterizes the rates of convergence. He constructs algorithms of solution for SPN on the basis of the Lagrangian relaxation and proves that with probability one these algorithms give the length of SMT. The paper includes a number of interesting results on the strategy of solution of the SPN.

In [15] Bertsimas considers a probabilistic version of the minimum spanning tree problem, where the probability p_i of presence of each vertex i is given, and suggests an approximate (suboptimal) algorithm for this problem.

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