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## **Graphs and Other Combinatorial Topics**

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This Teubner-Text comprises a substantial part of the papers and communications presented at the Third Czechoslovak Symposium on Graph Theory, held in Prague 1982.

Most topics of the contributions concern modern areas in graph theory, such as probabilistic methods, Ramsey theory, colouring, Hamiltonian problems. Other topics: combinatorics; algebra and graphs; computer-oriented graph theory.

# ENUMERATION OF ACYCLIC SUPERTOURNAMENTS OF A FINITE, LABELED, ACYCLIC DIGRAPH

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Let  $G$  be a finite, labeled, acyclic digraph. In this paper we consider a tournament, a complete oriented graph. A tournament of which a given digraph is a spanning subgraph is called a supertournament of this digraph.

The purpose here is to give a summary of the main results in solving the following problem 1 and some equivalent problems 2-4 (For the detailed proofs of this results see /1/).

Problem 1. Calculation of the number  $t(G)$  of all different acyclic supertournaments of a finite, labeled, acyclic digraph  $G$ .

Let  $V(G)$  and  $E(G)$  be the sets of vertices and arcs of a digraph  $G$ , respectively. Let  $\bar{G}$  be the transitive closure and  $\hat{G}$  the basis digraph of the digraph  $G$ , respectively. The number of elements of a set  $X$  we denote by  $|X|$ .

L e m m a 1.  $(\bar{G}_1 = \bar{G}_2) \vee (\hat{G}_1 = \hat{G}_2) \implies t(G_1) = t(G_2)$ .

Let  $r \in E(G)$ . We write  $G-r$  and  $G/\bar{r}$ , respectively, for a digraph resulting from the digraph  $G$  by canceling the arc  $r$  and by changing the orientation of  $r$ , respectively.

L e m m a 2.  $\forall r \in E(G): t(G-r) = t(G) + t(G/\bar{r})$ .

Let  $od_G(x)$  and  $id_G(x)$  be the outdegree and the indegree of the vertex  $x \in V(G)$ , respectively. We call  $L$  a circuit of an acyclic digraph  $G$  ( $L \subseteq G$ ). The vertices  $b \in V(L)$  and  $e \in V(L)$ , respectively, are the initial and the terminal vertices of  $L$  if  $id_L(b) = 0$  and  $od_L(e) = 0$ , respectively.

By changing the orientation of a special subset of the arcs of  $L$  the circuit  $L$  can be transformed into a cycle. The

arcs of this subset are called arcs of negative orientation. The remaining arcs of  $L$  are called positive oriented. Now we label the arcs of  $L$  as follows (cf. fig.1). Starting with any terminal vertex  $e \in V(L)$  and then continuing with the negative orientation we label with  $1, 2, \dots, m$  all positive oriented arcs. Analogously, starting with  $e \in V(L)$  and then following the positive orientation we label with  $-1, -2, \dots, n$  all negative oriented arcs.

Here  $m$  and  $n$  are the numbers of the positive and negative oriented arcs of  $L$ , respectively.

Let  $L_i^*$  be the circuit resulting from  $L$  by changing the orientation of  $|i|$  arcs of  $L$  labeled  $1, 2, \dots, i$  if  $i \geq 0$  and  $-1, -2, \dots, i$  if  $i < 0$ .

Whereas  $L_i$  is the digraph originating from  $L_i^*$  by canceling the arc with the label  $i$ . The digraph  $G/L_i$  results from  $G$  the exchange of  $L$  for  $L_i$ .

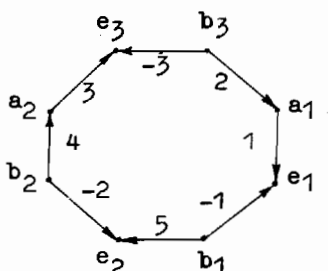


Fig.1. Circuit  $L$ .

**Theorem 1.** (Method of canceling a circuit)

$$\forall L \subseteq G: \quad t(G) = \frac{1}{2} \sum_{i=-n}^m (-1)^{i+1} t(G/L_i).$$

**Theorem 2.** For any finite, labeled, acyclic digraph  $G$  the problem of the calculation of  $t(G)$  can be reduced to the case of finite, labeled, oriented, rooted trees using the method of canceling circuits.

Let  $T$  denote a finite, labeled, oriented, rooted tree.

$\forall x \in V(T)$  let  $T_x$  be the maximal subtree with the root  $x$  in  $T$ .

**Theorem 3.**  $t(T) = |V(T)|! / \prod_{x \in V(T)} |V(T_x)|$ .

Let  $T_{n,m}$  denote a regular, oriented tree of degree  $n$

with height  $m$ .

C o r o l l a r y.

$$t(T_{n,m}) = (n-1)^{\frac{n^m-1}{n-1}} (n^{\frac{n^m-1}{n-1}})! \bigg/ \prod_{i=1}^m (n^{m-i+1} - 1)^{n^i}.$$

Let  $T/G$  denote an oriented, rooted, spanning tree of a weakly connected digraph  $G$ . Let  $I_n \triangleq \{1, 2, \dots, n\}$ .

T h e o r e m 4. Let  $G$  a finite, labeled, acyclic digraph.

$$(G = \bigsqcup_{i=1}^k G_i \mid G_i \text{ - weakly connected component of } G) \wedge$$

$$\wedge ((\forall i \in I_k) (\exists T/G_i \mid \hat{G}_i = T/G_i)) \implies$$

$$\implies t(G) = |V(G)|! \bigg/ \prod_{i=1}^k \prod_{x \in V(G_i)} |V(T_x/G_i)|.$$

Two families of upper bounds.

L e m m a 3.  $(G_{\mathbb{K}} \subseteq G \subseteq G^{\mathbb{K}}) \wedge (V(G_{\mathbb{K}}) = V(G) = V(G^{\mathbb{K}})) \implies$

$$\implies t(G^{\mathbb{K}}) \leq t(G) \leq t(G_{\mathbb{K}}).$$

T h e o r e m 5. Let  $G$  be a finite, labeled, acyclic, weakly connected digraph with exactly one initial vertex. Then

$$\forall T/G: t(G) \leq |V(G)|! \bigg/ \prod_{x \in V(G)} |V(T_x/G)|.$$

**Theorem 6.** Let  $G$  be a finite, labeled, acyclic digraph.

$$(G = \bigcup_{i=1}^k G_i) \wedge (\forall \{T/G_i/\}_{i \in I_k}) \Rightarrow \\ \Rightarrow t(G) \leq |V(G)|! / \prod_{i=1}^k \prod_{x \in V(G_i)} |V(T_x/G_i/)|.$$

We denote:

$$N(G) \triangleq \{X \subseteq V(G) \mid (\forall x, y \in X) ((x, y) \notin E(G) \wedge (y, x) \notin E(G))\},$$

$$q(G) \triangleq \max_{X \in N(G)} |X|.$$

**Theorem 7.** Let  $G$  be a finite, labeled, acyclic, weakly connected digraph with exactly one initial vertex  $b \in V(G)$ . Then for an arbitrary minimal Dilworth's decomposition of  $G$ , i.e.

$$V(G) = \bigcup_{i=1}^{q(G)} V(C_i), \quad \forall i \neq j: V(C_i) \cap V(C_j) = \emptyset,$$

where the  $C_i$  are paths in  $\bar{G}$ , we have

$$t(G) \leq \frac{(|V(G)| - 1)!}{(|V(C_p)| - 1)!} / \prod_{\substack{i=1 \\ i \neq p}}^{q(G)} |V(C_i)|!$$

Now we present three problems equivalent to problem 1.

**Problem 2.** Calculation of the number  $t(G)$  of all different topological sortings (/2/, p.365) or admissible labelings (/3/, p.49) of the vertices of a finite, labeled, acyclic digraph  $G$ , i.e.

$$t(G) = |\{f: I_n \rightarrow V(G) \mid ((f(i), f(j)) \in E(G) \Rightarrow (i < j))\}|, \\ \text{where } n = |V(G)|.$$

**Problem 3.** Calculation of the number  $t(R)$  of all different, admissible permutations of the partially ordered set  $I_n$ , i.e.

$$t(R) = |\{f: I_n \rightarrow I_n \mid f(i)Rf(j) \Rightarrow (i < j)\}|, \text{ where} \\ R \subseteq I_n \times I_n, R \cap E = \emptyset, R \cap R^{-1} = \emptyset, R^2 \subseteq R, E \triangleq \{(i, i) \mid i \in I_n\}. \\ (/4/, pp.39-40; /5/, /6/).$$

**Problem 4.** Calculation of the order dimension  $t(R)$  of a partial order  $R$   $X \times X$ , i.e.

$$t(R) = |\{L \subseteq X \times X \mid R \subseteq L\}|, \text{ where } L \cap E = \emptyset, \\ L \cap L^{-1} = \emptyset, L^2 \subseteq L, L \cup L^{-1} = X^2 \setminus E, E \triangleq \{(x, x) \mid x \in X\} (/7/; /8/, p.235).$$

#### R e f e r e n c e s

- /1/ Taraszow O.G. Zur Anzahl der kreisfreien Turniere, die Obergraph eines endlichen markierten kreisfreien Digraphen sind. EIK (to appear).
- /2/ Reingold E.M., Nievergelt J., Deo N. Combinatorial algorithms. Theory and Practice. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1977 (russ.transl., Mir, Moskva, 1980).
- /3/ Биркгоф Г., Барти Т. Современная прикладная алгебра. Пер. с англ., Мир, Москва, 1976.
- /4/ Танаев В.С., Шкурба В.В. Введение в теорию расписаний. Наука, Москва, 1975.
- /5/ Кислицин С.С. Конечные частично упорядоченные множества и соответствующие им множества перестановок. Мат. заметки 4, 5 /1968/, 511-518.
- /6/ Сидоренко А.Ф. Число допустимых линейных упорядочений частично упорядоченного множества как функция его графа несравнимости. Мат. заметки 29, I /1981/, 75-82.
- /7/ Dushnik B., Miller E.W. Partially ordered sets. Amer.J., Math., 63 (1941), 600-610.
- /8/ Ore O. Theory of graphs. Providence, Rhode Island, 1962 (russ.transl., Nauka, Moskva, 1968).